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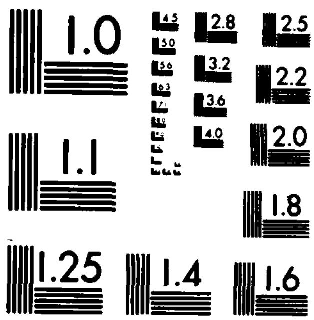

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Research Report CCS 454

THE ENTROPIC PENALTY APPROACH TO STOCHASTIC PROGRAMMING

by

A. Ben-Tal*

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by

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Key Words

Stochastic Programming, Penalty functions, Entropy, Statistical Information Theory, Saddle functions, Duality in Nonlinear Programming, Expected Utility Maximization, Constant Risk Aversion Utility functions, Nonsmooth Optimization, Convex Analysis.

Abbreviated Title

Stochastic Programming via Entropic Penalty

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ABSTRACT

A new decision-theoretic approach to Nonlinear Programming Problems with stochastic constraints is introduced. The Stochastic Program (SP) is replaced by a Deterministic Program (DP) in which a term is added to the objective function to penalize solutions which are not "feasible in the mean". The special feature of ~~our~~ approach is the choice of the penalty function P_E, which is given in terms of the relative entropy functional, and is accordingly called entropic penalty. It is shown that P_E has properties which make it suitable to treat stochastic programs. Some of these properties are derived via a dual representation of the entropic-penalty which also enable one to compute P_E more easily, in particular if the constraints in (SP) are stochastically independent. The dual representation is also used to express the Deterministic Problem (DP) as a saddle function problem. For problems in which the randomness occurs in the rhs of the constraints, it is shown that the dual problem of (DP) is equivalent to Expected Utility Maximization of the classical Lagrangian dual function of (SP), with the utility being of the constant-risk-aversion type. Finally, mean-variance approximations of P_E and the induced Approximate Deterministic Program are considered.

INTRODUCTION

Mathematical Programming problems with stochastic constraints,

(SP) $\inf\{g_0(x) : g(x,b) \geq a\}$,

dependening on a random vector b, are the subject of our investigation. A new decision-theoretic approach is suggested in the paper as a possible way to treat these stochastic programs. The approach is based on imitating the <u>penalty function method</u> of <u>deterministic Nonlinear Programming</u>. In this method the constrained problem is replaced by an unconstrained one, in which the new objective function has the property of "penalizing" (increasing the minimand) violations of the constraints. With an appropriate interpretation of "violation of constraints" in the stochastic case, and with an appropriate choice of the penalty function, to reflect the stochastic environment of the problem, we derive a deterministic problem (DP) replacing (SP):

(DP) $\inf\{g_0(x) + pP_E(x)\}$

where $p > 0$ is a penalty parameter, and P_E is our penalty function. This function is given in terms of the <u>relative entropy functional</u>, widely used in Statistical Information Theory, [5], [6].

If f_b is the generalized density of the random vector $b \in R^k$, and D_k is the set of all generalized densities f of random vectors $z \in R^k$ (all absolutely continuous with respect to a common nonnegative measure dt), then the relative entropy $I(f,f_b)$ between the random vectors z and b is

$$I(f,f_b) = \int f(t)\log\frac{f(t)}{f_b(t)}\,dt\ .$$

The penalty function is given by

$$P_E(x) = \inf_{f \in D_k} \{I(f,f_b): \int g(x,t)f(t)dt \geqslant a\}$$

and is called accordingly <u>entropic penalty</u>. The motivation for choosing P_E and the induced deterministic program (DP) is discussed in Chap.1. Properties of the entropic penalty, studied in Chap. 2, help further to demonstrate the appropriateness of using (DP). It is shown that $P_E(x)$ penalizes "violation of constraint in the mean", i.e. $P_E(x) = 0$ if $Eg(x,b) \geqslant a$ and $P_E(x) > 0$ otherwise. In this sence (DP) is a "relaxation" of the deterministic program

$$\inf\{g_0(x): Eg(x,b) \geqslant a\}$$

which can be recovered from (DP) by letting p be large enough. The latter program includes in particular the familiar <u>chance constraints problem</u> [2]. Another desirable property of P_E is that <u>surely infeasible solutions</u>, i.e. those x's that are infeasible for <u>any</u> realization of the random vector b, are excluded from (DP), since for those (and only those) $P_E(x) = \infty$. It is also shown that a greater "violation in the mean" of a constraint, results in a greater penalty.

Some of the above mentioned properties of the entropic penalty are derived from its definition, while other rely heavily on a <u>dual representation</u> of P_E, which also provides an easy way to compute it:

$$P_E(x) = \sup_{y \geq 0} \{y^T a - \log Ee^{y^T g(x,b)}\} .$$

The duality theory needed to obtain the dual expression is developed in Chap. 3. This representation can be further simplified , and for <u>independent constraints</u> (i.e. $g_i(x,b) = g_i(x,b_i)$ and the b_i's are independent random variables) it has an <u>explicit</u> representation in term of the function $\psi_i(t) = \log Ee^{y_i g_i(x,b_i)}$ and its derivative. The dual representation also enable us to express the deterministic problem (DP) as a saddle-value problem, and finally to demonstrate that (DP) is equivalent to the problem

$$\inf_x \sup_{y \geq 0} EU(\ell_b(x,y))$$

where U is the Constant Risk Aversion (CRA) utility function $U(t) = -e^{-(1/p)t}$ and $\ell_b(x,y)$ is the classical Lagrangian corresponding to (SP):

$$\ell_b(x,y) = g_0(x) - y^T(g(x,b) - a).$$

The important special case of (SP):

(SP-RHS) $\inf\{g_0(x): g(x) \leq b\}$

is thoroughly discussed in Chap. 4. The outstanding result which is obtained for such convex stochastic programs is the nature of the dual problem to the primal entropic-penalty program (DP); *The dual decision-maker is an expected utility maximizer, possessing a CRA type utility function U with an Arrow-Pratt risk indicator $(-U'/U'')$ equal to the reciprocal of the penalty parameter.* While in the deterministic case the dual problem is

$$\max_{y \geq 0} (\inf_x \ell_b(x,y)),$$

in the stochastic case our approach leads to the dual problem

$$\max_{y \geq 0} EU(\inf_x \ell_b(x,y)).$$

The Expected Utility Maximization is one of the fundamental approaches of Economics and Decision Theory under Uncertainty. The fact that (DP) generates such a sound dual is perhaps the most convincing argument in favor of the entropic penalty approach.

In Chap. 5 we obtain simple approximations of $P_E(x)$, in terms of the mean vector and the variance-covariance matrix of the random vector $g(x,b)$. The approximated entropic penalty $\hat{P}_E(x)$ is then given as the optimal value of a simple convex quadratic program with only nonnegativity constraints, or, for independent constraints, by an explicit formula involving $m_i(x) = Eg_i(x,b_i)$ and $\sigma_i^2(x)$ - the variance of $g_i(x,b_i)$:

$$\hat{P}_E(x) = \frac{1}{2}\Sigma\frac{1}{\sigma_i^2(x)}\left[\max(0,a_i - m_i(x))\right]^2 \quad .$$

For stochastic RHS Problems the approximations reduce to:

$$\hat{P}_E(x) = \sup_{y\geq 0}\{y^T(\mu - g(x)) - \frac{1}{2}\, y^T V y\}$$

where $\mu = Eb$ and V is the variance-covariance matrix of b. The approximation is exact if b is jointly Normal: $b \sim N(\mu,V)$.

Using these approximation in (DP) one obtains an Approximate Deterministic Problem (ADP):

$$\text{(ADP)} \quad \inf\{g_o(x) + p\hat{P}_E(x)\}.$$

As an illustration, for a stochastic RHS Problem with independent b_i's (having μ_i and variance σ_i^2) the Approximate Deterministic Problem is:

$$\text{(ADP)} \quad \inf\left\{g_o(x) + \frac{p}{2}\,\Sigma\,\frac{1}{\sigma_i^2}\left[\max(0,g_i(x) - \mu_i)\right]^2\right\} .$$

The latter program is similar to the one used in the classical penalty function method for the constrained (deterministic) problem

$$\inf\{g_o(x) : g(x) \leq \mu\}$$

except for the presence of the coefficients $1/\sigma_i^2$. The role of these, in the stochastic case, is to attribute smaller significance to "more ambiguous" constraints, i.e. those for which the rhs b_i has larger variance.

Problem (ADP) just mentioned, and a score of other problems occuring in the paper, give rise to interesting problems in Nonsmooth Optimization that may entail the use of numerical methods developed for such purposes, see e.g. [1], and [7].

As a general introduction to existing methods in Stochastic Programming, the reader is referred to the excellent review articles by Dempster (Part I in [3]) and Kall [4].

CHAPTER 1 - THE ENTROPIC PENALTY APPROACH

Consider the nonlinear programming problem

(SP) $\inf\{g_0(x): g(x,b) \geqslant a\}$,

where $x \in R^n$ is the decision vector; $b \in R^k$ and $a \in R^m$ are fixed parameters, and g is the vector-valued constraint function $g: R^n \times R^k \to R^m$. Let the feasible set be denoted by

$$S_b = \{x: g(x,b) \geqslant a\} .$$

Frequently (P) is converted to an unconstrained problem by adjoining to the objective function $g_0(x)$ a penalty function $P(x)$ and thus replacing (P) with

$$\inf\{g_0(x) + pP(x)\} \tag{1}$$

where $p > 0$ is a penalty parameter. The function $P(x)$ is generally a distance function measuring how far is x from the feasible set, i.e.

$$P(x) = \mathrm{dist}(x,S_b),$$

but it can also be given in terms of the distance between b and the set

$$S_x^{-1} = \{z: g(x,z) \geqslant a\} ,$$

i.e.

$$P(x) = \mathrm{dist}(b,S_x^{-1}) = \inf_z\{\mathrm{dist}(b,z): z \in S_x^{-1}\} .$$

Problem (1) becomes then

$$\inf_x\left\{g_0(x) + p \inf_z\{\mathrm{dist}(b,z): g(x,z) \geqslant a\}\right\} . \tag{2}$$

The original problem (P) is in fact a special case of problem (1) with

$$P(x) = \begin{cases} 0 & \text{if } \; g(x,b) \geqslant a \\ \infty & \text{otherwise} \end{cases}$$

or with finite-valued $P(\cdot)$ but with penalty parameter p very large. In other cases (1) (and hence (2)) can be viewed as a <u>relaxation</u> of (P).

Assume now (and henceforth in this paper) that the parameter vector b is <u>stochastic</u>, with distribution function $F_b(\cdot)$, absolutely continuous w.r.t. a nonnegative measure dt, and possessing a generalized density (Radon Nikodym derivative) $f_b(\cdot)$. Let $B \subset R^k$ be the support of b.

Looking back at problem (2), one should naturally think now of z as a <u>random</u> vector. Thus it remains to interpret two things: (a) the meaning of a "distance between two random variables" and (b) the meaning of "$g(x,z) \geqslant a$" when z is random. As for point (a) there is a classical answer, which is the fundamental concept in Statistical Information Theory (see e.g. the book by Kullback [5])

$$\text{dist}(b,z) = I(f_z,f_b) = \int_B f_z(t) \log \frac{f_z(t)}{f_b(t)}\, dt^* \quad .$$

The integral $I(f_z,f_b)$ is the so called <u>relative entropy</u> or <u>divergence</u>. It legitimacy as a "distance function" comes (among other things) from its well-known property

<u>Proposition 1</u>. $I(f_z,f_b) \geqslant 0$ and is equal to zero if and only if $f_z = f_b$ (a.e.).

---

* This is a short notation for

$$\iint \cdots \int_B f_z(t_1,\ldots,t_k) \log \frac{f_z(t_1,\ldots,t_k)}{f_b(t_1,\ldots,t_k)}\, dt_1,\ldots dt_k$$

As for the second point (b), we adopt the interpretation that "$g(x,z) \geq a$ holds in the mean", i.e.

$$E_z g(x,z) \geq a.$$

The result is a penalty function $P_E(\cdot)$, called entropic penalty, which is given by

$$P_E(x) = \inf_{f \in D_k} \left\{ \int_B f(t) \log \frac{f(t)}{f_b(t)}\, dt : \int_B g_i(x,t) f(t) dt \geq a_i, \quad i = i,\ldots,m \right\} \quad (3)$$

where D_k is the set of all generalized densities of random vectors $z \in R^k$, which are absolutely continuous w.r.t. the measure dt.

In terms of the entropic penalty, we introduce the Determinstic Primal (DP) problem as a surrogate for the Stochastic Primal (SP) problem:

$$\text{(DP)} \quad \inf_x \{g_o(x) + pP_E(x)\} .$$

Let us note then if x is such that f_b itself satisfies the constraint in (3), i.e.

$$E_b g_i(x,b) \geq a_i, \qquad i = 1,\ldots,m, \qquad (4)$$

then the optimal density is f_b itself, and by Proposition 1 it follows that $P(x) = 0$. At the same time, if x is such that (4) is violated then $P(x) > 0$. Therefore, (DP) is a relaxation of the following, more naive, deterministic replacement of (SP), namely

$$\inf\{g_o(x) : E_b g(x,b) \geq a\} . \qquad (5)$$

As a concrete example, let $g(x,b)$ be chosen as

$$g(x,b) = \begin{cases} 1 & \text{if } g(x) \leqslant b \\ \\ 0 & \text{if } g(x) \not\leqslant b \end{cases} \tag{6}$$

and let $a = 1-\alpha$ $(0 < \alpha < 1)$. Problem (5) becomes the well-known <u>Chance Constrained program</u> (see [2]):

$$\text{(CC)} \quad \inf\{f(x) : \Pr(g(x) \leqslant b) \geqslant 1-\alpha\} \ .$$

The corresponding Deterministic Primal, which in this case is denoted (CCDP),

$$\text{(CCDP)} \quad \inf_x \{g_0(x) + p \cdot \inf_{f \in D_k} \{I(f,f_b) : \int_{g(x)}^{\infty} f(t)dt \geqslant 1-\alpha\}\} \ ,$$

penalizes violations of the chance constraints. (CC) can be recovered from (CCDP) by choosing p sufficiently large.

CHAPTER 2 - PROPERTIES OF THE ENTROPIC-PENALTY

In this Chapter some important properties of P_E *are derived.* Additional properties will be discussed in Chap. 3 as well. These properties demonstrate the appropriateness of using the entropic penalty for solving Stochastic Programming problem.

<u>Proposition 2:</u>

$$P_E(x)\begin{cases} = 0 & \text{if } \; E_b g_i(x,b) \geq a_i \quad \forall i \\ = \infty & \text{if } \; \bar{g}_i(x) = \sup\limits_{b\in B} g_i(x,b) < a_i \; \text{ for some } i \\ \text{positive and finite} & - \; \text{otherwise} \end{cases}$$

<u>Proof:</u> By Proposition 1, $P_E(x) \geq 0$ with equality if and only if the optimal f is equal to f_b (a.e), this is possible if and only if f_b is feasible i.e. $E_b g_i(x,b) \geq a_i$, $\forall i$. It remains to show that $P_E(x) = \infty$ if and only if $\bar{g}_i(x) < a_i$ for some i. The latter means that the constraints in (3) are infeasible, implying $P_E(x) = \infty$. That the opposite is also true (i.e., $P_E(x) = \infty$ implies (3) is infeasible) follows from Theorem 1(b) in Chap. 3.

□

The proposition demonstrates that P_E is a <u>penalty</u> function for the constraints $Eg(x,b) \geq a$ and a <u>barrier</u> function for the constraints $\bar{g}(x) \geq a$. The Deterministic Primal problem (DP) can be rewritten as

$$\text{(DP)} \qquad \inf_x \{g_0(x) + \rho P_E(x) : \bar{g}(x) \geq a\} \; .$$

Note that $\bar{g}(x) \not\geq a$ means that x is not feasible for the original (SP) problem <u>for any realization of b</u>, and exactly these <u>surely infeasible</u> solutions are ruled out by (DP)

The next two results concern independent constraints. We say that $g(x,b) \geq a$ are <u>independent constraints</u> if the components $\{b_i\}$ of b are independent random variables, and if, for each i, the i-th constraint depends only on b_i, i.e., $k = m$ and

$$g_i(x,b) = g_i(x,b_i) \quad .$$

We make it clear that in this case, the set D_k in (3) is the set of all generalized densities of random vectors $z \in R^k$, with independent components, so $f(t_1,\ldots,t_m) = \prod_{i=1}^{m} f_i(t_i)$.

<u>Proposition 3</u>: For independent constraints, P_E is given by

$$\begin{cases} P_E(x) = \sum_{i=1}^{m} P_E^i(x) \quad \text{where} \\ P_E^i(x) = \inf_{f_i \in D_1} \left\{ \int f_i(t_i) \log \frac{f_i(t_i)}{f_{b_i}(t)} dt_i : \int g_i(x,t_i) f_i(t_i) dt_i \geq a_i \right\} \end{cases} \tag{7}$$

<u>Proof</u>: The result follows from the well-known additivity property of the relative entropy for independent random variable ([5] Th. 2.1).

□

The proposition expresses the useful fact that, whenever the constraints are independent, the penalty for the system of constraints equals to the sum of penalties for the individual constraints.

We say that <u>x^1 is less feasible than x^2 for the i-th constraint</u> (in the mean) if

$$E_b g_i(x^1,b) - a_i < E_b g_i(x^2,b) - a_i.$$

Proposition 4: Let the constraints of (SP-RHS) be independent. If x^1 is less feasible than x^2 for the i-th constraints, then

$$P_E^i(x^1) > P_E^i(x^2) \quad .$$

□

The next results concerns Stochastic RHS problems:

(SP-RHS) $\inf\{g_0(x): g(x) \leq b\}$.

This is a special case of (SP) with

$$g(x,b) = b - g(x), \quad a = 0 \quad . \tag{8}$$

Proposition 5: For a Stochastic RHS problem

$$P_E(x) = \inf_{f \in D_k} \{I(f,f_b): \int tf(t)dt \geq g(x)\} \quad .$$

If (SP-RHS) is a convex program, then $P_E(x)$ is a convex function.

Proof: The equation (9) follows from a simple substitution of (8) in (3). The convexity result will be proved in Chap. 4 via a dual expression for P_E, from which the conclusion of Proposition 4 follows too. □

A convexity result holds also for the chance constrained problem (CC). The proof is also postponed to Chap. 3 (see Remark 1, following Example 1).

Proposition 6: If (CC) has independent and concave constraints (i.e. for each i and each b_i, $Pr(g_i(x) \leq b_i)$ is a concave function of x) then $P_E(x)$ is a convex function.

CHAPTER 3 - A DUAL REPRESENTATION OF P_E AND A SADDLE FUNCTION REPRESENTATION OF (DP)

The value of the entropic penalty function P_E at a given point x, is the optimal value of the extremal problem

$$(E) \qquad \inf_{f \in D_k} I(f) = \int f(t) \log \frac{f(t)}{f_b(t)} \, dt$$

subject to (10)

$$\int g_i(x,t) f(t) dt \geq a_i, \qquad i = 1,\ldots,m \ .$$

We will write this shortly as

$$P_E(x) = \inf(E).$$

By constructing a dual problem for (E), say (H), a dual representation of P_E will follow:

$$P_E(x) = \sup(H).$$

To construct (H) we first need an auxiliary result.

Lemma 1: Let $c(t)$ be a given positive summable function:

$$\int c(t) dt = C < \infty \ .$$

Then

$$\inf_{f \in D_k} \int f(t) \log \frac{f(t)}{c(t)} \, dt = - \log \int c(t) dt. \tag{11}$$

Proof: Use the identity

$$\int f(t) \log \frac{f(t)}{c(t)} dt = \int f(t) \log \frac{f(t)}{c(t)/C} dt - \log C. \tag{12}$$

Now, since $c(t)/C$ is a density, it follows from Proposition 1 that the first term in the rhs of (12) is minimized by $f(t) = c(t)/C$ and its optimal value is zero, so the infimal value of the lhs of (12) is $-\log C$, as claimed.

□

We now form the Lagragian of problem (E), $L\colon D_k \times R_+^m \to R$ with values

$$L(f,y) = I(f) - \sum_{i=1}^{m} y_i \int g_i(x,t)f(t)dt + a^T y \ .$$

The dual objective function is

$$g(y) = \sup_{f \in D_k} L(f,y)$$

or, more explicitly

$$g(y) = \inf_{f \in D_k} \{I(f) - \Sigma y_i \int g_i(x,t)f(t)dt + a^T y\}$$

$$= \inf_{f \in D_k} \left\{\int \left[\log\left(\frac{f(t)}{f_b(t)}\right) - \Sigma y_i g_i(x,t)\right] f(t)dt\right\} + a^T y$$

$$= \inf_{f \in D_k} \left\{\int \log\left(\frac{f(t)}{f_b(t)\exp(\Sigma y_i g_i(x,t))}\right) f(t)dt\right\} + a^T y$$

$$= a^T y - \log \int f_b(t) e^{\Sigma y_i g_i(x,t)} dt, \quad \text{(by Lemma 1)}.$$

So, the dual of (E) is

$$\text{(H)} \quad \sup_{y \geq 0} \{a^T y - \log \int f_b(t) e^{\Sigma y_i g_i(x,t)} dt\} \ .$$

<u>Theorem 1</u>: [Duality Theory for (E)-(H).]

(a) If (E) is feasible then inf(E) is attained and

min(E) = sup(H).

(b) $\sup(H) < \infty$ if and only if (E) is feasible

(c) sup(H) is attained if there exists a density f in D_k satisfying the constraints (10) strictly, in which case

min(E) = max(H).

Moreover, if $f^* \in D_k$ solves (E) and $y^* \geqslant 0$ solves (H) then

$$f^*(t) = \frac{f_b(t)e^{-\Sigma y_i^* g_i(x,t)}}{\int f_b(t)e^{-\Sigma y_i^* g_i(x,t)}dt} \quad \text{(a.e.)}$$

Proof: We set problem (E) as a convex problem, in an appropriate vector space, with finitely many linear constraints, as follows. Let $M(B)$ be the linear space of real-valued finite regular Borel measure (rBm) on B. Let dt be a nonnegative rBm on B. For $\mu \in M(B)$, which is absolutly continuous w.r.t. dt we denote by $\frac{d\mu}{dt}$ its Radon-Nikodym derivative. Whenever $\mu \in S$ (the convex subset of probability measures) we call $f(t) = \frac{d\mu}{dt}$ a (generalized) density. Let

$$J(\mu) = \begin{cases} \int_B f(t)\log\frac{f(t)}{f_b(t)}\,dt & \text{if } \mu \text{ is an abs. cont. probability measure, and } f = \frac{d\mu}{dt} \\ \infty & \text{otherwise} \end{cases}$$

and consider the linear operator $A: \mu(B) \to R^m$

$$\mu \xrightarrow{A} \begin{pmatrix} \int g_1(x,t)d\mu \\ \vdots \\ \int g_m(x,t)d\mu \end{pmatrix} \tag{13}$$

Then, problem (E) amounts to

$$\inf\{J(\mu): A\mu \geqslant a\} . \tag{14}$$

Now, (H) is just the Lagrangian dual of (14) (which here coincides with the Fenchel-Rockafellar dual [11]) and most of the results in the theorem follow from standard duality relations (e.g. [10], [11], and [8]). Thus, the fact that the dual (H) has only nonnegativity constraints $y \geq 0$ (and hence satisfying the strongest constraint qualification) implies lack of duality gap and attainment of the primal infimum. Part (c) is just the usual dual statement. As for part (b), the implication

$$\text{(E)feasible} \Rightarrow \sup(H) < \infty$$

follows from weak duality. Thus, only the reverse implication

$$\text{(E)infeasible} \Rightarrow \sup(H) = \infty \tag{15}$$

is exceptional here and needs special care.
The feasible set of (E) is

$$A\mu \geq a \qquad T\mu = 1 \qquad \mu \text{ nonnegative} \tag{16}$$

where A is the linear operator (13), and T is the linear function

$$\mu \xrightarrow{T} \int d\mu \quad .$$

Using a duality theorem for linear program in vector spaces (e.g. [8], Theorem 3.13.8, p. 68), it follows that the infeasibility of (16) is equivalent to the feasibility of

$$A^*y + T^*v \leq 0, \quad y'a + v' > 0 \quad y \in R^m_+, \quad v \in R \quad . \tag{17}$$

Here $A^*: R^m \to C(B)$, $T^*: R \to C(B)$ are the adjoints of A and T respectively: $A^*y = \Sigma y_i g_i(x,t)$; $T^*v = v$ (a constant function in $C(B)$ — the linear space of continuous function on B — the dual space of $M(B)$). So, (17) implies that

$$\begin{cases} \exists \bar{y} \geq 0, \quad \bar{v} \in R \quad \text{such that} \\ \qquad\qquad \Sigma \bar{y}_i g_i(x,t) + \bar{v} \leq 0, \quad \bar{y}'a + \bar{v} > 0. \end{cases} \tag{18}$$

Now, using the identity

$$0 = v - \log e^{v}$$

it is easy to see that the dual program (H) is equivalent to

$$\sup_{y \geq 0, v \in R} \{y'a + v - \log \int f_b(t) e^{\Sigma y_i g_i(x,t) + v} dt\} . \tag{19}$$

By taking $\bar{y}$, $\bar{v}$ from (18), and $M > 0$ arbitrary large, it is seen that the sup in (19) is made arbitrary large by choosing $y = M\bar{y}$, $v = M\bar{v}$, i.e. $\sup(H) = \infty$.

□

From Theorem 1 we obtain a dual representation of the entropic penalty function, which is much simpler than the primal expression given by (3):

$$P_E(x) = \sup_{y \geq 0} \{y^T a - \log \int f_b(t) e^{\sum_{i=1}^{m} y_i g_i(x,t)} dt\} . \tag{20}$$

This representation is a key factor in deriving important facts (some mentioned already in Chap. 2) about P_E and about the dual problem of (DP). As an "appetizer" we obtain the explicit expression of P_E for independent chance constraints.

<u>Example 1</u>: Problem (CC) with independent constraints is

$$\inf\{g_0(x): \Pr(g_i(x) \leq b_i) \geq 1-\alpha_i, \quad i = 1,\dots,m\}$$

and the corresponding (CCDP) problem is

$$\inf\{g_0(x) + \rho \Sigma P_E^i(x)\}.$$

By (20):

$$P_E^i(x) = \sup_{0\leq y\in R} \{y(1-\alpha_i) - \log \int f_b(t)e^{yg_i(x,t)} dt\} . \tag{21}$$

Recalling from (16) that

$$g_i(x,t) = \begin{cases} 1 & \text{if } g_i(x) \leq t \\ 0 & \text{otherwise} \end{cases}$$

we get from (21), in term of the cumulative distribution function F_i of b_i,

$$P_E^i(x) = \sup_{y\geq 0} \{y(1-\alpha_i) - \log[(1-F_i(g_i(x)))e^y + F_i(g_i(x))]\} . \tag{22}$$

By simple calculus, the maximizing y is y_i^* given by

$$y_i^* = \begin{cases} \log\left[\dfrac{(1-\alpha_i)F_i(g_i)}{\alpha_i(1-F_i(g_i))}\right] & \text{if } F(g_i(x)) \geq \alpha_i \\ 0 & \text{if } F(g_i(x)) \leq \alpha_i \end{cases} .$$

Substituting y_i^* in (22) yields

$$P_E^i(x) = \begin{cases} 0 & \text{if } F_i(g_i(x)) \leq \alpha_i \text{ i.e. } Pr(g_i(x) \leq b_i) \geq 1-\alpha_i \\ \alpha_i \log \dfrac{\alpha_i}{F_i(g_i(x))} + (1-\alpha_i)\log\left(\dfrac{1-\alpha_i}{1-F_i(g_i(x))}\right) & \text{if } F_i(g_i(x)) \geq \alpha_i \end{cases} . \tag{23}$$

Remark 1: The function $h_i(t) = \alpha_i \log \frac{\alpha_i}{t} + (1-\alpha_i) \log \frac{1-\alpha_i}{1-t}$ is convex and increasing for $0 < \alpha_i \leq t < 1$ and $h(\alpha_i) = 0$. If $F_i(g_i(x))$ is convex (i.e. if $Pr(g_i(x) \leq b_i)$ is concave in x) then $h_i(F_i(g_i(x))$ is convex for x such that $\alpha_i \leq F_i(g_i(x))$. This proves that $P_E^i(x)$ is convex since by the above and (23):

$$P_E^i(x) = h_i(\max(\alpha_i, F_i(g_i(x))).$$

The objective function in (20), in term of which P_E is computed, is $y^T a - \psi(y)$ where

$$\psi(y) = \log E_b e^{y^T g(x,b)} . \tag{24}$$

If the random vector $g(x,b)$ is <u>nondegenerate</u> (i.e. $\forall y \neq 0$, $y^T g(x,b)$ is not a degenerate univariate random variable), then $\psi(y)$ is strictly convex, as follows from the following:

<u>Lemma 2:</u> If Z is a nondegenerate random vector in R^m, then the function

$$\phi(y) = \log E_Z e^{y^T Z}$$

is strictly convex in y.

<u>Proof</u>: Consider the function $h(t_1,t_2) = t_1^\lambda t_2^{1-\lambda}$ $(0 < \lambda < 1)$. It is strictly concave for $t_1 > 0$, $t_2 > 0$, $t_1 \neq t_2$, so by Jensen inequality

$$E(t_1^\lambda t_2^{1-\lambda}) < E(t_1)^\lambda E(t_2)^{1-\lambda} .$$

Put $t_1 = e^{y_1^T Z}$, $t_2 = e^{y_2^T Z}$, then

$$E_Z\left(e^{\lambda y_1^T Z} \cdot e^{(1-\lambda) y_2^T Z}\right) < \left(E_Z e^{y_1^T Z}\right)^\lambda \left(E_Z e^{y_2^T Z}\right)^{1-\lambda}$$

or, taking log,

$$\log E_Z e^{(\lambda y_1 + (1-\lambda) y_2)^T Z} < \lambda \log E_Z e^{y_1^T Z} + (1-\lambda) \log E_Z e^{y_2^T Z}$$

which proves the strict convexity of $\phi(y)$.

□

We will derive still another expression of P_E in term of the conjugate function ψ^* of ψ i.e.

$$\psi^*(u) = \sup_y (u^T y - \psi(y)).$$

Proposition 7:

$$P_E(x) = \inf_{u \geq a} \psi^*(u) \tag{25}$$

where ψ^* is the conjugate of the strictly convex function ψ, given in (24). Moreover, if the expectation $E_b e^{y^T g(x,b)}$ is finite for every y then

$$\psi^*(u) = u^T \nabla\psi^{-1}(u) - \psi(\nabla\psi^{-1}(u)) \tag{26}$$

where $\nabla\psi$ is the gradient vector of ψ, i.e. the i-th component of $\nabla\psi$ is

$$[\nabla\psi(y)]_i = \frac{E_b g_i(x,b) e^{y^T g(x,b)}}{E_b e^{y^T g(x,b)}} \quad .$$

Proof: By (20)

$$P_E(x) = \sup_{y \geq 0} \{y^T a - \psi(y)\} \quad . \tag{27}$$

The Lagrangian dual of the problem in the rhs of (27) is easily seen to be

$$\inf_{v \geq 0} \psi^*(a+v)$$

and with change of variables $u = a+v$ one obtains (25). The strict convexity of ψ follows from Lemma 2, and the finiteness assumption implies that ψ is also smooth. Hence $\nabla\psi$ is a strictly monotone mapping and ψ^* coincides with its Legendre Transform, which is the rhs of (26) (see [12], Chap. 26).

□

<u>Example 2:</u> Consider the Stochastic RHS problem (SP-RHS) with b a jointly Normal random vector, with mean vector μ and covariance matrix V (positive definite since b is assumed nondegenerate). Then a direct computation shows that here (20) becomes the quadratic program:

$$P_E(x) = \sup_{y \geq 0} \{y^T(g(x) - \mu) - \frac{1}{2} y^T V y\} ,$$

while (27) is the dual quadratic program:

$$P_E(x) = \inf_{u \geq 0} \{(g(x) - \mu - u)^T V^{-1}(g(x) - \mu - u)\} .$$

For a Stochastic Program with independent constraint a further simplification of the expression for P_E is possible. In fact, the infimum in (25) can be computed, and we get an explicit representation of P_E in terms of the conjugate function ψ_i^* of

$$\psi_i(y_i) = \log E_{b_i} e^{y_i g_i(x,b_i)} .$$

We use the following notations: for a function $h(t)$, $h: R \to R$ let

$$Dh = \frac{d}{dt} h, \quad D^{-1}h = \left(\frac{d}{dt} h\right)^{-1}$$

Let also

$$m_i(x) = E_{b_i} g_i(x,b_i).$$

<u>Proposition 8:</u> For (SP), with independent constraints

$$P_E(x) = \sum_{i=1}^{m} \psi_i^*(\max(m_i(x),a_i) \tag{28}$$

where

$$\psi_i^*(t) = tD^{-1}\psi_i(t) - \psi_i(D^{-1}\psi_i(t)). \tag{29}$$

Moreover, $\psi_i^*(t)$ is a strictly increasing function for $t > m_i(x)$.

<u>Proof</u>: By Proposition 3, $P_E(x) = \Sigma P_E^i(x)$, therefore we have to show that

$$P_E^i(x) = \psi_i^*(\max(m_i(x),a_i)).$$

Now, from Proposition 7:

$$P_E^i(x) = \inf_{u \geq a_i} \psi_i^*(u_i)$$

where ψ_i^* is exactly given by (29). The function ψ_i^* is strictly convex and simple calculus shows that

$$P_E^i(x) = \inf_{u \geq a_i} \psi_i^*(u) = \psi_i^*(\max(D^{-1}\psi^*(0),a_i)) \ . \tag{30}$$

But it is a well known fact of conjugate functions that $D^{-1}\psi^* = D\psi$, so

$$D^{-1}\psi^*(0) = D\psi(0) = E_{b_i} g_i(x,b_i) = m_i(x). \tag{31}$$

Using this in (30), the desired expression for P_E^i is obtained. To prove the last statement of the proposition, note that from (31)

$$0 = D\psi^*(m_i(x))$$

and since ψ^* is strictly convex, this implies

$$D\psi^*(t) > 0, \quad \text{for} \quad t > m_i(x),$$

which establishes the claimed monotonicity.

□

Remark 2: The last statement of Proposition 8 and (28) provides a proof for Proposition 4.

Consider the saddle function

$$k(x,y) = g_o(x) + p(y'a - \log E_b e^{y^T g(x,b)}). \tag{32}$$

Then, by the dual expression (20) of P_E, we see that the Deterministic Primal problem (DP) becomes

$$\text{(DP)} \inf_x \sup_{y \geq 0} k(x,y).$$

An equivalent program will be generated if we use another saddle function

$$\mathcal{L}(x,y) = - e^{-\frac{1}{p} k(x,y/p)}$$

obtained from k by one-to-one transformations of its domain and range. Now, a little algebra shows that

$$\mathcal{L}(x,y) = -E_b e^{-\frac{1}{p}\{g_o(x) - y^T(g(x,b)-a)\}},$$

thus, we proved:

Theorem 2: The Deterministic Primal problem (DP), derived via the entropic penalty approach, is equivalent to the saddle-function problem

$$\text{(DP-EU)} \quad \inf_x \sup_{y \geq 0} EU(\ell_b(x,y))$$

where $U(\cdot)$ is the constant-risk-aversion utility function $U(t) = -e^{-\frac{1}{p}t}$ (or any positive affine transformation of it) and where $\ell_b(x,y)$ is the classical Lagrangian corresponding to the original (SP) problem, i.e.

$$\ell_b(x,y) = g_o(x) - y^T(g(x,b) - a).$$

□

CHAPTER 4 - THE DUAL PROBLEM OF (DP) FOR STOCHASTIC RHS PROGRAMS

In this section we treat exclusively the problem

(SP-RHS) $\inf\{g_0(x): g_i(x) \leq b_i, \quad i = 1,\dots,m\}$.

This is a specialization of the general (SP) problem with $g(x,b) = b - g(x)$ and $a = 0$. The expression for the entropic penalty, is given in (9). From the results of Chap. 3, dual representations of P_E are, by (20) and Proposition 7:

$$P_E(x) = \sup_{y\geq 0} \{y^T g(x) - \log E_b e^{y^T b}\} \tag{34}$$

or

$$\begin{cases} P_E(x) = \inf\limits_{u\geq g(x)} \phi^*(u) \quad \text{where} \quad \phi(y) = \log E_b e^{y^T b}, \quad \text{and} \\ \phi^*(u) = u^T \nabla\phi^{-1}(u) - \phi(\nabla\phi^{-1}(u)). \end{cases} \tag{35}$$

If the b_i's are independent random variables with $E(b_i) = \mu_i$, then by Proposition 8:

$$\begin{cases} P_E(x) = \sum\limits_{i=1}^{m} \phi_i^*(\max(g_i(x),\mu_i) \quad \text{where} \quad \phi_i(y) = \log E_{b_i} e^{y_i b_i} \quad \text{and} \\ \phi_i^*(t) = tD^{-1}\phi_i(t) - \phi_i(D^{-1}\phi_i(t)). \end{cases} \tag{36}$$

Note that by (34), if $g(x)$ is convex so is $P_E(x)$, as was claimed in Proposition 5. From Proposition 2 we also know that

$$P_E(x) \begin{cases} = 0 & \text{if} \quad g(x) \leq \mu \\ \infty & \text{if} \quad g(x) \nleq b_{max} \\ \text{positive and finite} & - \quad \text{otherwise} \end{cases}$$

Here b_{max} is the vector whose i-th component is the right extreme value of the support of b_i. Therefore, the Deterministic Primal problem is here a relaxation of the problem

$$\inf_x \{g_0(x): g(x) \leq \mu\}$$

and it rules out surely infeasible solutions, i.e. those x's for which $g(x) \not\leq b_{max}$

We have already computed P_E for the case of joint Normal random variables (Example 2). We add here two more examples for (SP-RHS) with independent b_i's.

<u>Example 3</u>: (Independent Poisson variates). Let the b_i's be independent random variable each having a Poisson distribution with parameter (mean) λ_i, so

$$f_{b_i}(k) = \frac{1}{k!} e^{-\lambda_i} \lambda_i^k, \quad k = 0,1,2,\ldots\ldots .$$

The function $\phi_i(\cdot)$ in (36) is the log of the moment generating function, so

$$\phi_i(y) = \lambda_i(e^y - 1).$$

The derivative is $D\phi_i(y) = \lambda_i e^y$, the inverse is $D^{-1}\phi_i(t) = \log(t/\lambda_i)$ and thus by (36):

$$\phi_i^*(t) = t \log(t/\lambda_i) - t + \lambda_i .$$

Note that ϕ_i^* is a convex and strictly increasing function for $t > \lambda_i$, as anticipated by Proposition 8.

The final expression for P_E is by (36):

$$P_E(x) = \sum_{i=1}^{m} \lambda_i \left\{ \left(1 + \frac{(g_i(x)-\lambda_i)_+}{\lambda_i}\right) \log\left(1 + \frac{(g_i(x)-\lambda_i)_+}{\lambda_i}\right) - \frac{(g_i(x)-\lambda_i)_+}{\lambda_i} \right\}^{\dagger}$$

Example 4: (Independent Gamma variates). Let each b_i have a Gamma distribution with parameters λ_i and r_i, i.e. the density is

$$f_{b_i}(t) = \frac{\lambda_i}{\Gamma(r_i)} (\lambda_i t)^{r_i - 1} e^{-\lambda_i t}, \qquad t > 0.$$

The mean is $\mu_i = E(b_i) = r_i/\lambda_i$, and the moment generating function is $(1 - y/\lambda_i)^{-r_i}$, $(y < \lambda_i)$. Therefore here

$$\phi_i(y) = -r_i \log(1-y/\lambda_i) = -r_i \log(1-y\mu_i/r_i), \qquad y < r_i/\mu_i \;;$$

$$D\phi(y) = \frac{\mu_i r_i}{r_i - \mu_i y} \;;\; D^{-1}\phi(t) = (r_i/\mu_i)(1-\mu_i/t), \qquad t > \mu_i \;;$$

$$\phi_i^*(t) = r_i[t/\mu_i - 1 - \log(t/\mu_i)], \qquad t > \mu_i .$$

We obtain finally from (36):

$$P_E(x) = \Sigma r_i \left\{ \frac{(g_i(x)-\mu_i)_+}{\mu_i} - \log\left(1 + \frac{g_i(x)-\mu_i)_+}{\mu_i}\right)\right\} .$$

Note that for the Gamma distribution, the variance (σ_i^2) of b_i's is $\sigma_i^2 = r_i/\lambda_i^2 = \mu_i^2/r_i$, so $r_i = \mu_i^2/\sigma_i^2$ and P_E is given in terms of the mean and variance by

$$P_E(x) = \Sigma \frac{\mu_i^2}{\sigma_i^2} \left\{ \frac{(g_i(x)-\mu_i)_+}{\mu_i} - \log\left(1 + \frac{(g_i(x)-\mu_i)_+}{\mu_i}\right)\right\} . \tag{37}$$

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† For a real number α we denote $\alpha_+ = \max(\alpha, 0)$

In terms of the saddle function (32), which here becomes:

$$k(x,y) = g_0(x) + p(y^T g(x) - \log E_b e^{y^T b}) , \tag{38}$$

The Primal Deterministic Problem $\inf\{g_0(x) + pP_E(x)\}$ is

(DP-RHS) $\quad \inf_x \sup_{y\geq 0} k(x,y).$

We define the <u>Dual Deterministic Problem</u> (DD-RHS) corresponding to (DP-RHS) by

(DD-RHS) $\quad \sup_{y\geq 0} \inf_x k(x,y).$

Thus, the dual objective function is

$$h(y) = \inf_x k(x,y) \qquad (k(x,y) \text{ given in } (38))$$

and the dual problem is

(DD-RHS) $\quad \sup_{y\geq 0} h(y).$ †

The key issue, of course, regarding the dual pair (DP-RHS) and (DD-RHS), is the <u>lack of duality gap</u>, which here corresponds to the <u>existence of saddle value for k</u>, i.e. the validity of

$$\inf_x \sup_{y\geq 0} k(x,y) = \sup_{y\geq 0} \inf_x k(x,y). \tag{39}$$

In this connection we make use of two conditions which guarantee (39) for a general convex-concave saddle function $k(x,y)$.

<u>Condition 1</u>: (Stoër [13] Corollary 2.13) "The $\inf_x \sup_{y\geq 0} k(x,y)$ is attained

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† The problem may include implicitly more constraints on y coming from the requirement $E_b e^{y^T b} < \infty$.

and $k(x,\cdot)$ is strictly concave".

Condition 2: (Rockafellar [9], Theorem 8(I), see in particular the Example on page 173) "No nonzero $y_o \geqslant 0$ has the property

$$y_o^T \nabla_y k(x,y) \geqslant 0 \qquad \forall (x \in R^n,\ y > 0)."$$

We now establish a minimax theorem for $k(x,y)$ in (38).

Theorem 3: Let (SP-RHS) be a convex aprogram, (i.e. g_o and g_i, $i = 1,\ldots,m$ are convex functions), and consider the saddle function in (38):

$$k(x,y) = g_o(x) + p(y^T g(x) - \log E_b e^{y^T b}) \ .$$

Then, either one of the following two conditions

(i) $\inf\limits_{x} \sup\limits_{y \geq 0} k(x,y)$ is attained

(ii) $\exists \hat{x} \in R^n$ such that $g(\hat{x}) < b_{max}$,

implies the existance of a saddle value for k, i.e. the validity of (39).

Proof: The convexity of g_o, and all g_i $(i = 1,\ldots,m)$ implies that $k(\cdot,y)$ is convex for every $y \geqslant 0$. From Lemma 2 we know that

$$\psi(y) = \log E_b e^{y^T b}$$

is strictly convex, hence $k(x,\cdot)$, in (38), is strictly concave. Therefore, condition (i) in the Theorem suffices to imply condition 1 of Stoër. Condition 2 of Rockafellar reduces here to the nonexistence of a nonzero $y_o \geqslant 0$ such that

$$y^o \left[g(x) - \frac{E(b e^{y^T b})}{E e^{y^T b}} \right] \geqslant 0 \quad \forall (x \in R^n,\ y > 0). \tag{40}$$

This is clearly satisfied if

$$\exists \hat{x} \quad \text{and} \quad \hat{y} > 0 \quad \text{such that} \quad g(\hat{x}) < \frac{E(be^{\hat{y}^T b})}{Ee^{\hat{y}^T b}} = \nabla\psi(\hat{y}) \quad . \tag{41}$$

To show that condition (ii) implies (41) it suffices to demonstrate that

$$\bar{b}_i = (b_{max})_i < \sup_{0 \le y \in R^m} \frac{\partial}{\partial y_i} \psi(y) \quad . \tag{42}$$

Let $\psi_i(y_i) = \psi(0,0,\ldots,y_i,\ldots 0)$, $i = 1,2,\ldots,m$, i.e.

$$\psi_i(y_i) = \log Ee^{y_i b_i} .$$

Note that

$$\sup_{0 \le y_i \in R} \psi_i'(y_i) < \sup_{0 \le y \in R^m} \frac{\partial}{\partial y_i} \psi(y), \quad \forall i$$

hence to prove (42) it suffices to prove that

$$\bar{b}_i < \sup_{0 \le y_i \in R} \psi_i'(y_i) = \sup_{0 \le y_i} \left(\frac{E(b_i e^{y_i b_i})}{Ee^{y_i b_i}} \right) \tag{43}$$

For this purpose consider a special case of problem (E) in Chap. 3 with a single random variable b_i, and with $a_i = \bar{b}_i$, $g_i(x,t) = t$ and a single constraint (the i-th), i.e.

$$(E_i) \qquad \inf_{f \in D_1} \{I(f,f_{b_i}) : \int tf(t)dt > \bar{b}_i\}$$

The dual program is (see Chap. 3)

$$(H_i) \qquad \sup_{0 \le y \in R} \{\bar{b}_i y - \log Ee^{y_i b_i}\} = \sup_{y \ge 0} \{\bar{b}_i y - \psi_i(y)\}$$

Program (E_i) is clearly feasible (take $f(t) = 1$ for $t = \bar{b}_i$ and $f(t) = 0$ otherwise) and hence, by Theorem 1, $\sup(H_i) < \infty$. Now $\psi_i(y)$ is convex and $\psi_i(0) = 0$, hence by the gradient inequality

$$0 = \psi_i(0) \geq \psi_i(y) - y\psi_i'(y)$$

and we get

$$\infty > \sup(H_i) = \sup_{y\geq 0}\{\bar{b}_i y - \psi_i(y)\} \geq \sup_{y\geq 0}\{\bar{b}_i y - y\psi_i'(y)\}$$

$$= \sup_{y\geq 0}\{y(\bar{b}_i - \psi_i'(y))\} \quad .$$

For the latter to be finite for $y \to \infty$ it is necessary that $\bar{b}_i \leq \lim_{y\to\infty}\psi_i'(y)$, but since ψ_i' is strictly increasing (a derivative of the strictly convex function ψ_i) this is the same as (43), and the proof is completed.

$\square$

Remark 3: Condition (ii), which guarantees the lack of duality gap for (DP-RHS) and (DD-RHS), is extremely mild. Indeed, if it does not hold, then for almost all realizations of b, the original (SP) problem is infeasible. If such ill-posed stochastic programs are rules out, then the entropic penalty Deterministic Primal always induces an equivalent dual program. We shall see shortly what is the meaning of this dual program.

Remark 4: Condition (ii) implies in fact a stronger saddle-value result than (39), namely

$$\inf_x \sup_{y\geq 0} k(x,y) = \max_{y\geq 0} \inf_x k(x,y).$$

i.e. the supremum of the dual objective function $h(y)$ is attained. (see [9]). Condition (i), which assumes that for some $\bar{x},\bar{y} > 0$

$$\inf_{x} \sup_{y\geq 0} k(x,y) = k(\bar{x},\bar{y})$$

implies in fact attainment of the dual saddle value at the same point, i.e.

$$\sup_{y\geq 0} \inf_{x} k(x,y) = \max_{y\geq 0} \min_{x} k(x,y) = k(\bar{x},\bar{y}). \quad \text{(See [13].)}$$

Remark 5: Stochastic Programs satisfying condition (i) or (ii) of Theorem 3 will be called well-posed.

Let $\ell_b(x,y)$ be the classical Lagrangian corresponding to (SP-RHS):

$$\ell_b(x,b) = g_o(x) + y^T(g(x) - b)$$

and consider the constant-risk-aversion (CRA) utility function

$$U(t) = -e^{-\frac{1}{p}t} \quad \text{(or any positive affine transformation of it).}$$

It follows from Theorem 2, that the primal problem (DP-RHS) is equivalent to

(DP-EU) $\quad \inf_{x} \sup_{y\geq 0} EU(\ell_b(x,y)).$

Therefore, the dual problem (DD-RHS) is equivalent to

(DD-EU) $\quad \sup_{y\geq 0} \inf_{x} EU(\ell_b(x,y)).$

To get the full meaning of this dual problem we first prove

Lemma 3:

$$\inf_{x} EU(\ell_b(x,y) = EU(\inf_{x} \ell_b(x,y)) . \tag{44}$$

Proof:

$$EU(\inf_x \ell_b(x,y)) = E \inf_x U(\ell_b(x,y)) \qquad \text{since } U \text{ is monotone increasing}$$

$$= E \inf_x \left\{-e^{-\frac{1}{p}(g_o(x)+y^T g(x)-y^T b)}\right\} =$$

$$= E\left(e^{\frac{1}{p}y^T b}\right)\inf_x \left\{-e^{-\frac{1}{p}(g_o(x)+y^T g(x))}\right\} =$$

$$= \inf_x\left(Ee^{\frac{1}{p}y^T b}\right)\left(-e^{-\frac{1}{p}(g_o(x)+y^T g(x))}\right) = \inf_x EU(\ell_b(x,y)) \ .$$

□

Recall that for a non-stochastic problem, the classical Lagrangian dual is the concave program

$$\sup_{y\geq 0} \ h(y) = \inf_x \ell_b(x,y) \quad .$$

From the lemma we observe that in the stochastic case, the dual problem (DD-EU) consists of maximizing the expected utility of the Lagrangian dual function with the utility function being of the CRA-type. More precisely, combining the results in Theorems 2,3 and the Lemma 3 we have actually proven:

Theorem 4: Consider a well-posed convex stochastic program (SP-RHS). Let (DP-RHS) be the corresponding entropic penalty Deterministic Primal and let (DD-RHS) be the corresponding Deterministic Dual. Then, (DD-RHS) is equivalent to the concave program

$$\text{(DD-EU)} \qquad \max_{y\geq 0} EU(h(y)); \qquad h(y) = \inf_x \ell_b(x,y)$$

where U is the CRA-utility function with the Arrow-Pratt risk indicator being equal to the reciprocal of the penalty parameter P.

□

CHAPTER 5 - MEAN-VARIANCE APPROXIMATIONS

We obtain in this section quadratic approximations of $P_E(x)$, for the general (SP) problem

$$(SP) \quad \inf\{g_0(x):\ g(x,b) \geq a\} \ .$$

For every fixed x, the random vector $g(x,b)$ is assumed non-degenerate, with mean vector

$$m(x) = Eg(x,b)$$

and (positive definite) variance-covariance matrix

$$V(x) = COV(g(x,b)) \ .$$

The variance vector (diagonal of $V(x)$) is denoted by $\sigma^2(x)$.

Recall from Chap. 3 that

$$P_E(x) = \sup_{y \geq 0} \{y^T a - \psi(y)\}$$

where

$$\psi(y) = \log Ee^{y^T g(x,b)} \ . \tag{45}$$

Now, straightforward calculations show that

$$\psi(0) = 0 \tag{46}$$

$$\nabla\psi(0) = m(x) \tag{47}$$

$$\nabla^2\psi(0) = V(x). \tag{48}$$

Hence, a second-order Taylor expansion of $\psi(y)$ yield the following approximation $\hat{P}_E(x)$ of $P_E(x)$; in terms of a concave quadratic program.

<u>Proposition 9</u>:

$$\hat{P}_E(x) = \sup_{y\geq 0} \{y^T[a-m(x)] - \frac{1}{2} y^T V(x) y\}$$

Another expression for the approximation $P_E(x)$ is given in terms of the following convex quadratic program.

<u>Proposition 10</u>:

$$\hat{P}_E(x) = \inf_{u\geq a} \{\frac{1}{2} (u-m(x))^T V(x)^{-1} (u-m(x))\} \quad .$$

<u>Proof</u>: By Proposition 7: $P_E(x) = \inf_{u\geq a} \psi^*(u)$ where ψ^* is the conjugate function of ψ in (45). Thus it remains to show that a second order approximation $\hat{\psi}^*$ of ψ^* is

$$\hat{\psi}^*(u) = \frac{1}{2} (u-m(x))^T V(x)^{-1} (u-m(x)). \tag{49}$$

Since the gradient of ψ and its conjugate are inverse operators, i.e. $\nabla\psi^* = \nabla\psi^{-1}$, (see [12], Chap. 26) it follows from (47) that

$$\nabla\psi^*(m(x)) = 0 \tag{50}$$

and so, by (26) and (46), also

$$\psi^*(m(x)) = 0 \quad . \tag{51}$$

Now

$\nabla^2\psi^* = \nabla(\nabla\psi^*) = \nabla[(\nabla\psi)^{-1}] = [\nabla^2\psi(\nabla\psi^{-1})]^{-1}$, by the Inverse Function Theorem, in particular then, by (47), (48):

$$\nabla^2\psi^*(m(x)) = V(x)^{-1} \quad . \tag{52}$$

A second order Taylor expansion of ψ^*:

$$\hat{\psi}^*(u) = \psi(m(x)) + (u-m(x))^T \nabla\psi^*(m(x)) + \frac{1}{2}(u-m(x))^T \nabla^2\psi^*(m(x))(u-m(x)) \tag{53}$$

indeed agrees with (49) by substituting (50)-(52) in (53). □

Remark 6: If the random vector $g(x,b)$ is jointly Normal then

$$Ee^{y^T g(x,b)} = \exp(y^T m(x) - y^T V(x) y)$$

so, $\psi(y)$ is quadratic, and hence coincides with its Taylor series approximation $\hat{\psi}(y)$. The same is true of course for ψ^*. Therefore, the approximations $\hat{P}_E(x)$ in Propositions 9 and 10 are exact.

If the constraints $g_i(x,b) \geq a_i$ are independent we can use Proposition 8 and the Taylor expansion (49) to obtain:

Proposition 11: For (SP) with independent constraints, a second order apprcximations $\hat{P}_E(x)$ of $P_E(x)$ is

$$\hat{P}_E(x) = \frac{1}{2} \Sigma \frac{1}{\sigma_i^2(x)} [(a_i - m_i(x))_+]^2$$

where

$$m_i(x) = E_{b_i} g_i(x,b_i), \quad \sigma_i^2(x) = \text{variance of } g_i(x,b_i).$$

□

For stochastic RHS programs (SP-RHS) the above approximation simplifies as follows: let $\mu = Eb$, denote by V the variance-covariance matrix of b, and by σ_i^2 the variance of b_i. Then, by Proposition 9,

$$\hat{P}_E(x) = \sup_{y \geq 0} \{y^T(\mu - g(x)) - \frac{1}{2} y^T V y\} \quad .$$

When $b \sim N(\mu,V)$ the approximation is exact; compare with Example 2.

The approximate entropic penalty $\hat{P}_E$ induces an Approximate Deterministic Primal problem to (SP):

(ADP) $$\inf_x \{g_0(x) + \rho \hat{P}_E(x)\} \quad .$$

By Proposition 9, this problem can be stated in terms of the saddle function

$$\hat{k}(x,y) = g_0(x) + p[y^T(a-m(x)) - \frac{1}{2} y^T V(x) y]$$

as

$$\text{(ADP)} \qquad \inf_x \sup_{y \geq 0} \hat{k}(x,y) \, .$$

In the case of independent constraints, an explicit representation of (ADP), based on Proposition 11 is

$$\text{(ADP)} \qquad \inf_x \left\{ g_0(x) + p \, \Sigma \frac{1}{\sigma_i^2(x)} [(a_i - m_i(x))_+]^2 \right\}$$

This, further simplifies for a Stochastic RHS problem to (see Prop. 11):

$$\inf_x \left\{ g_0(x) + \frac{p}{2} \Sigma \frac{1}{\sigma_i^2} [(g_i(x) - \mu_i)_+]^2 \right\} \qquad (54)$$

Remark 7: If the variance of b_i (σ_i^2) is large, then as seen from (54), the contribution of the i-th constraint to the penalty $\hat{P}_E$ is small. Therefore, "ambiguous constraints" are effectively ignored in the Approximate Deterministic Primal. The quantity $1/\sigma_i^2$ thus serves as a "built-in" penalty parameter for the i-th constraint.

Remark 8: The approximate penalty function $\hat{P}_E$ does not necessarily possess the property that surely infeasible solutions are ruled out. Therefore, in (ADP) one should add the constraints $\bar{g}(x) \geqslant a$ (see Chap. 2). For (SP-RHS) the added constraint are $g(x) \leqslant b_{max}$.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A new decision-theoretic approach to Nonlinear Programming Problems with stochastic constraints is introduced. The Stochastic Program (SP) is replaced by a Deterministic Program (DP) in which a term is added to the objective function to penalize solutions which are not "feasible in the mean." The special feature of our approach is the choice of the penalty function P_E, which is given in terms of the relative entropy functional, and is accordingly called <u>entropic penalty</u>. It is shown that P_E has

20. ABSTRACT (continued)

properties which make it suitable to treat stochastic programs. Some of these properties are derived via a dual representation of the entropic-penalty which also enable one to compute P_E more easily, in particular if the constraints in (SP) are stochastically independent. The dual representation is also used to express the Deterministric Problem (DP) as a saddle function problem. For problems in which the randomness occurs in the rhs of the constraints, it is shown that the dual problem of (DP) is equivalent to Expected Utility Maximization of the classical Lagrangian dual function of (SP), with the utility being of the constant-risk-aversion type. Finally, mean-variance approximations of P_E and the induced Approximate Deterministic Program are considered.

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